Regular nilpotent elements and quantum groups

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Abstract

We suggest new realizations of quantum groups $U_q(\mathfrak{g})$ corresponding to complex simple Lie algebras, and of affine quantum groups. These new realizations are labeled by Coxeter elements of the corresponding Weyl group and have the following key feature: The natural counterparts of the subalgebras $U(\mathfrak{n})$, where $\mathfrak{n} \subset \mathfrak{g}$ is a maximal nilpotent subalgebra, possess non–singular characters.

Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{b}$ a Borel subalgebra, and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ its nilradical. We denote by (,) the Killing form on $\mathfrak{g}$. An element $f \in \mathfrak{n}$ is called *regular nilpotent* if its centralizer in $\mathfrak{g}$ is of minimal possible dimension. Any regular nilpotent element $f \in \mathfrak{n}$ defines a character χ of the opposite nilpotent subalgebra $\overline{\mathfrak{n}} = [\overline{\mathfrak{b}}, \overline{\mathfrak{b}}]$, where $\overline{\mathfrak{b}}$ is the opposite Borel subalgebra. Naturally, the character χ extends to a character of the universal enveloping algebra $U(\overline{\mathfrak{n}})$. Recall that $U(\overline{\mathfrak{n}})$ is generated by the positive simple root generators $X_i^+, i = l, \ldots, rank\ \mathfrak{g}$ of the Chevalley basis associated with the

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pair $(\mathfrak{g}, \overline{\mathfrak{b}})$. On these generators the character χ takes values $c_i \neq 0$, $\chi(X_i^+) = c_i$. Conversely, any character of this form determines a regular nilpotent element in $\mathfrak{n}$.

Regular nilpotent elements are of great importance in the structure theory of Lie algebras and in its applications. In particular, there is a relation between regular nilpotent elements of a complex semisimple Lie algebra and Coxeter elements of the corresponding Weyl group [9]. Other applications of regular nilpotent elements include the theory of Whittacker modules in representation theory of semisimple Lie algebras [10], the integrability of the Toda lattice [11], and the remarkable realization of the center of the universal enveloping algebra of a complex simple Lie algebra as a Hecke algebra [10]. This provides a motivation to look for counterparts of regular nilpotent elements in the theory of quantum groups.

Let $U_q(\mathfrak{g})$ be the quantum group associated with a complex simple Lie algebra $\mathfrak{g}$, and let $U_q(\mathfrak{n})$ be the subalgebra of $U_q(\mathfrak{g})$ corresponding to the nilpotent Lie subalgebra $\mathfrak{n} \subset \mathfrak{g}$. $U_q(\mathfrak{n})$ is generated by simple positive root generators of $U_q(\mathfrak{g})$ subject to the q–Serre relations. It is easy to show that $U_q(\mathfrak{n})$ has no nondegenerate characters (taking nonvanishing values on all simple root generators)! Our first main result is the family of new realizations of the quantum group $U_q(\mathfrak{g})$, one for each Coxeter element in the corresponding Weyl group. The counterparts of $U(\mathfrak{n})$, which naturally arise in these new realizations of $U_q(\mathfrak{g})$, do have non–singular characters. Thus, we get proper quantum counterparts of $U(\mathfrak{n})$ and of its non–singular characters. As a byproduct, we derive an interesting formula for the Caley transform of a Coxeter element.

Next, we generalize our consideration to the case of affine Lie algebras. Similar to the finite-dimensional situation, the subalgebra $U_q(\mathfrak{n}((z)))$ in the affine quantum group $U_q(\widehat{\mathfrak{g}})$ naturally corresponding to $\mathfrak{n}((z))$ has no characters taking nonvanishing values on the quantum counterparts of the loop generators of $\mathfrak{n}((z))$. Again, we introduce new realizations of $U_q(\widehat{\mathfrak{g}})$, labeled by Coxeter elements, such that the natural counterparts of $U(\mathfrak{n}((z)))$ acquire such characters. Our realizations are variations of the Drinfeld's 'new realization' of affine quantum groups [3]. The perspective application of our construction is the Drinfeld-Sokolov reduction for affine quantum groups.

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1 Non–singular characters and finite–dimensional quantum groups

In this section we construct quantum counterparts of the principal nilpotent Lie subalgebras of complex simple Lie algebras and of their non–singular characters.

We follow the notation of [7]. Let $\mathfrak{h}^*$ be an l–dimensional complex vector space, $a_{ij}, i, j = 1, \ldots, l$ a Cartan matrix of finite type , $\Delta \in \mathfrak{h}^*$ the corresponding root system, and $\{\alpha_1, ..., \alpha_l\}$ the set of simple roots. Denote by W the Weyl group of the root system Δ, and by $s_1, ..., s_l \in W$ reflections corresponding to simple roots. Let $d_1, \ldots, d_l$ be coprime positive integers such that the matrix $b_{ij} = d_i a_{ij}$ is symmetric. There exists a unique non–degenerate W–invariant scalar product $(,)$ on $\mathfrak{h}^*$ such that $(\alpha_i, \alpha_j) = b_{ij}$.

Let $\mathfrak{g}$ be the complex simple Lie algebra associated to the Cartan matrix a_{ij}. Denote by $\mathfrak{n} \subset \mathfrak{g}$ the principal nilpotent subalgebra generated by the simple positive root generators of the Chevalley basis.

Definition 1 *A character $\chi : \mathfrak{n} \to \mathbb{C}$ is called non–singular if and only if it takes non-vanishing values on all simple root generators of $\mathfrak{n}$.*

Note that any non–singular character is equivalent (up to a Lie algebra automorphism of $\mathfrak{n}$) to χ_0 which takes value 1 on each simple root generator. Any character of $\mathfrak{n}$ naturally extends to a character of the associative algebra $U(\mathfrak{n})$. It is our goal to construct quantum counterparts of the algebra $U(\mathfrak{n})$ and of the non–singular character χ_0.

Let q be a complex number, $0 < |q| < 1$. Put $q_i = q^{d_i}$. We consider the simply–connected rational form $U_q^R(\mathfrak{g})$ of the quantum group $U_q(\mathfrak{g})$ [2], Section 9.1. This is an associative algebra over $\mathbb{C}$ with generators $X_i^\pm, L_i, L_i^{-1}, i = 1, \ldots, l$ subject to the relations:

$L_iL_j = L_jL_i \ , \ L_iL_i^{-1} = L_i^{-1}L_i = 1,$

$L_iX_j^{\pm}L_i^{-1} = q_i^{\pm\delta_{i,j}}X_j^{\pm},$

$X_i^+X_j^- - X_j^-X_i^+ = \delta_{i,j}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$

$K_i = \prod_{j=1}^{l} L_j^{a_{ji}},$

and the q–Serre relations: (1)

$\sum_{r=0}^{1-a_{ij}}(-1)^r \left[\begin{array}{c} 1-a_{ij} \\ r \end{array}\right]_{q_i} (X_i^{\pm})^{1-a_{ij}-r}X_j^{\pm}(X_i^{\pm})^r = 0, \ i \neq j,$

where

$\left[\begin{array}{c} m \\ n \end{array}\right]_q = \frac{[m]_q!}{[n]_q![n-m]_q!}, \ [n]_q! = [n]_q \ldots [1]_q, \ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$

The elements $X_i^{\pm}$ correspond to the simple positive (negative) root generators. We would like to show that the algebra spanned by $X_i^+, i = 1, \ldots, l$ does not admit characters which take nonvanishing values on all generators X_i^+, except for the case of $U_q(sl(2))$ when q-Serre relations do not appear.

Suppose, χ is such a character, and $\chi(X_i) = c_i$. The q-Serre relations are homogeneous and, hence, one can put $c_i = 1$ for all i without loss of generality. By applying the character χ to the q-Serre relations one obtains a family of identities,

$$\sum_{r=0}^{1-a_{ij}}(-1)^r \left[\begin{array}{c} 1-a_{ij} \\ r \end{array}\right]_{q_i} = 0, \ i \neq j. \tag{2}$$

We claim that some of these relations fail for the quantized universal enveloping algebra $U_q^R(\mathfrak{g})$ of any simple Lie algebra $\mathfrak{g}$, with the exception of $\mathfrak{g} = sl(2)$. In a more general setting , relations (2) are analysed in the following lemma.

Lemma 1 *The only rational solutions of equation*

$$\sum_{k=0}^{m}(-1)^k \left[\begin{array}{c} m \\ k \end{array} \right]_t t^{kc} = 0, \tag{3}$$

where t is a complex number, $0 < |t| < 1$, are of the form

$$c = -m+1, -m+2, \ldots, m-2, m-1. \tag{4}$$

Proof. According to the q–binomial theorem [6],

$$\sum_{k=0}^{m}(-z)^k \left[\begin{array}{c} m \\ k \end{array} \right]_t = \prod_{p=0}^{m-1}(1 - t^{m-1-2p}z). \tag{5}$$

Put $z = t^c$ in this relation. Then the l.h.s of (5) coincides with the l.h.s. of (3). Now (5) implies that $c = m-1-2p, p = 0, \ldots, m-1$ are the only rational solutions of (3).

Now we return to identities (2). Any Cartan matrix contains at least one off-diagonal element equal to -1. Then, $m = 1 - a_{ij} = 2$ and $c = \pm 1$, and lemma 1 implies that some of identities (2) are false for any simple Lie algebra, except for $sl(2)$. Hence , subalgebras of $U_q^R(\mathfrak{g})$ generated by X_i^+ do not possess non–singular characters.

It is our goal to construct subalgebras of $U_q^R(\mathfrak{g})$ which resemble the subalgebra $U(\mathfrak{n}) \subset U(\mathfrak{g})$ and possess non–singular characters. Denote by S_l the symmetric group of l elements. To any element $\pi \in S_l$ we associate a Coxeter element s_π by the formula $s_\pi = s_{\pi(1)} \ldots s_{\pi(l)}$. For each Coxeter element s_π we define an associative algebra F_q^π generated by elements $e_i,\ i = 1, \ldots l$ subject to the relations :

$$\sum_{r=0}^{1-a_{ij}}(-1)^r q^{rc_{ij}^\pi} \left[\begin{array}{c} 1-a_{ij} \\ r \end{array} \right]_{q_i} (e_i)^{1-a_{ij}-r} e_j (e_i)^r = 0,\ i \neq j, \tag{6}$$

where $c_{ij}^\pi = \left(\frac{1+s_\pi}{1-s_\pi}\alpha_i, \alpha_j\right)$ are matrix elements of the Caley transform of s_π in the basis of simple roots.

Proposition 2 *The map $\chi_q^\pi : F_q^\pi \to \mathbb{C}$ defined on the generators by $\chi_q^\pi(e_i) = 1$ is a character of the algebra F_q^π.*

To show that χ_q^π is a character of F_q^π it is sufficient to check that the defining relations (6) belong to the kernel of χ_q^π ,i.e.

$$\sum_{r=0}^{1-a_{ij}} (-1)^r q^{rc_{ij}^\pi} \left[\begin{array}{c} 1-a_{ij} \\ r \end{array} \right]_{q_i} = 0,\ i \neq j. \tag{7}$$

As a preparation for the proof of proposition 2 we study the matrix elements of the Caley transform of s_π which enter the definition of F_q^π.

Lemma 3 *The matrix elements of* $\frac{1+s_\pi}{1-s_\pi}$ *are of the form :*

$$\left(\frac{1+s_\pi}{1-s_\pi} \alpha_i, \alpha_j \right) = \varepsilon_{ij}^\pi b_{ij}, \tag{8}$$

where

$$\varepsilon_{ij}^\pi = \begin{cases} -1 & \pi^{-1}(i) < \pi^{-1}(j) \\ 0 & i = j \\ 1 & \pi^{-1}(i) > \pi^{-1}(j) \end{cases} \tag{9}$$

Proof. (compare [1] , Ch. V , §6 , Ex. 3).

First we calculate the matrix of the Coxeter element s_π with respect to the basis of simple roots. We obtain this matrix in the form of the Gauss decomposition of the operator s_π.

Let $z_{\pi(i)} = s_\pi \alpha_{\pi(i)}$. Recall that $s_i(\alpha_j) = \alpha_j - a_{ji}\alpha_i$. Using this definition the elements $z_{\pi(i)}$ may be represented as:

$$z_{\pi(i)} = y_{\pi(i)} - \sum_{k \geq i} a_{\pi(k)\pi(i)} y_{\pi(k)},$$

where

$$y_{\pi(i)} = s_{\pi(1)} \dots s_{\pi(i-1)} \alpha_{\pi(i)}. \tag{10}$$

Using matrix notation we can rewrite the last formula as follows:

$$\begin{gathered} z_{\pi(i)} = (I+V)_{\pi(k)\pi(i)} y_{\pi(k)}, \\ \text{where } V_{\pi(k)\pi(i)} = \begin{cases} a_{\pi(k)\pi(i)} & k \geq i \\ 0 & k < i \end{cases} \end{gathered} \tag{11}$$

To calculate the matrix of the operator s_π with respect to the basis of simple roots we have to express the elements $y_{\pi(i)}$ via the simple roots. Applying the definition of simple reflections to (10) we can pull out the element $\alpha_{\pi(i)}$ to the right:

$$y_{\pi(i)} = \alpha_{\pi(i)} - \sum_{k<i} a_{\pi(k)\pi(i)} y_{\pi(k)}.$$

Therefore

$$\alpha_{\pi(i)} = (I+U)_{\pi(k)\pi(i)} y_{\pi(k)} \text{ , where } U_{\pi(k)\pi(i)} = \begin{cases} a_{\pi(k)\pi(i)} & k<i \\ 0 & k \geq i \end{cases}$$

Thus

$$y_{\pi(k)} = (I+U)^{-1}_{\pi(j)\pi(k)} \alpha_{\pi(j)}. \tag{12}$$

Summarizing (12) and (11) we obtain:

$$s_\pi \alpha_i = \left((I+U)^{-1}(I-V)\right)_{ki} \alpha_k. \tag{13}$$

This implies:

$$\frac{1+s_\pi}{1-s_\pi}\alpha_i = \left(\frac{2I+U-V}{U+V}\right)_{ki} \alpha_k. \tag{14}$$

Observe that $(U+V)_{ki} = a_{ki}$ and $(2I+U-V)_{ij} = -a_{ij}\varepsilon^\pi_{ij}$. Substituting these expressions into (14) we get :

$$\left(\frac{1+s_\pi}{1-s_\pi}\alpha_i, \alpha_j\right) = -(a^{-1})_{kp}\varepsilon^\pi_{pi} a_{pi} b_{jk} = \tag{15}$$

$$-d_j a_{jk}(a^{-1})_{kp}\varepsilon^\pi_{pi} a_{pi} = \varepsilon^\pi_{ij} b_{ij}. \tag{16}$$

This concludes the proof of the lemma.

Proof of proposition 2 Identities (7) follow from lemma 1 for $t = q_i, \ \ m = 1 - a_{ij}, \ \ c = \varepsilon^\pi_{ij} a_{ij}$ since set of solutions (4) always contains $\pm(m-1)$.

Motivated by relations (6) we suggest new realizations of the quantum group $U_q^R(\mathfrak{g})$, one for each Coxeter element s_π. Let $U_q^\pi(\mathfrak{g})$ be the associative algebra over $\mathbb{C}$ with generators $e_i, f_i, L_i^{\pm 1} i = 1, \dots l$ subject to the relations:

$$
\begin{aligned}
&L_iL_j = L_jL_i, \quad L_iL_i^{-1} = L_i^{-1}L_i = 1 \\
&L_ie_jL_i^{-1} = q_i^{\delta_{i,j}}e_j, \quad L_if_jL_i^{-1} = q_i^{-\delta_{i,j}}f_j \\
&e_if_j - q^{c^\pi_{ij}}f_je_i = \delta_{i,j}\tfrac{K_i-K_i^{-1}}{q_i-q_i^{-1}}, \\
&K_i = \textstyle\prod_{j=1}^l L_j^{a_{ji}}, \qquad\qquad (17) \\
&\textstyle\sum_{r=0}^{1-a_{ij}}(-1)^r q^{rc^\pi_{ij}} \left[\begin{array}{c} 1-a_{ij} \\ r \end{array}\right]_{q_i} (e_i)^{1-a_{ij}-r}e_j(e_i)^r = 0, \ i \neq j, \\
&\textstyle\sum_{r=0}^{1-a_{ij}}(-1)^r q^{rc^\pi_{ij}} \left[\begin{array}{c} 1-a_{ij} \\ r \end{array}\right]_{q_i} (f_i)^{1-a_{ij}-r}f_j(f_i)^r = 0, \ i \neq j.
\end{aligned}
$$

It follows that the map $\tau^\pi_q : F^\pi_q \to U^\pi_q(\mathfrak{g}); \quad e_i \mapsto e_i$ is a *natural* embedding of F^π_q into $U^\pi_q(\mathfrak{g})$. From now on we identify F^π_q with the subalgebra in $U^\pi_q(\mathfrak{g})$ generated by $e_i, i = 1, \dots l$.

Theorem 4 *For every integer–valued solution $n_{ij} \in \mathbb{Z}, \ i,j = 1,\dots,l$ of equations*

$$d_in_{ji} - d_jn_{ij} = c^\pi_{ij} \tag{18}$$

there exists an algebra isomorphism $\psi_{\{n\}} : U^\pi_q(\mathfrak{g}) \to U^R_q(\mathfrak{g})$ defined by formulas:

$$
\begin{aligned}
&\psi_{\{n\}}(e_i) = q_i^{-n_{ii}}\textstyle\prod_{p=1}^l L_p^{n_{ip}}X_i^+, \\
&\psi_{\{n\}}(f_i) = \textstyle\prod_{p=1}^l L_p^{-n_{ip}}X_i^-, \qquad\qquad (19) \\
&\psi_{\{n\}}(L_i) = L_i.
\end{aligned}
$$

Proof is provided by direct verification of defining relations (17). The most nontrivial part is to verify deformed q–Serre relations (6). The defining relations of $U^R_q(\mathfrak{g})$ imply the following relations for $\psi_{\{n\}}(e_i)$,

$$\sum_{k=0}^{1-a_{ij}}(-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array} \right]_{q_i} q^{k(d_in_{ji}-d_jn_{ij})}\psi_{\{n\}}(e_i)^k\psi_{\{n\}}e_j\psi_{\{n\}}(e_i)^{1-a_{ij}-k}=0, \tag{20}$$

for any $i \neq j$. Now using equation (18) we arrive to relations (6).

Remark 1 *The general solution of equation (18) is given by*

$$n_{ji} = \frac{1}{2}(\varepsilon_{ij}a_{ij} + \frac{s_{ij}}{d_i}),. \tag{21}$$

where $s_{ij} = s_{ji}$. In order to show that integer solutions, $n_{ij} \in \mathbb{Z}$, exist, we choose $s_{ij} = b_{ij}$. Then, $n_{ji} = \frac{1}{2}(\varepsilon_{ij}+1)a_{ij}$, and this is an integer.

We call the algebra $U_q^\pi(\mathfrak{g})$ the Coxeter realization of the quantum group $U_q^R(\mathfrak{g})$ corresponding to the Coxeter element s_π. The subalgebra F_q^π and the character χ_q^π are quantum counterparts of $U(\mathfrak{n})$ and of the non–singular character χ_0 , respectively.

2 Non–singular characters and affine quantum groups

In this section we suggest new realizations of affine quantum groups labeled by Coxeter elements, similar to those described in the previous section for finite–dimensional quantum groups.

Let $\widehat{\mathfrak{g}} = \mathfrak{g}((z)) \dot{+} \mathbb{C}$ be the nontwisted affine Lie algebra corresponding to $\mathfrak{g}$ and let $\mathfrak{n}((z)) \subset \widehat{\mathfrak{g}}$ be the loop algebra of the nilpotent Lie subalgebra $\mathfrak{n} \subset \mathfrak{g}$.

Let χ be the character of $\mathfrak{n}$ which takes value 1 on all root generators of $\mathfrak{n}$. χ has a unique extension to the character $\widehat{\chi}$ of $\mathfrak{n}((z))$, such that $\widehat{\chi}$ vanishes on the complement $z^{-1}\mathfrak{n}[[z^{-1}]] + z\mathfrak{n}[[z]]$ of $\mathfrak{n}$ in $\mathfrak{n}((z))$.

It is our goal to define quantum counterparts of the algebra $U(\mathfrak{n}((z))) \subset U(\widehat{\mathfrak{g}})$ and of the character $\widehat{\chi}$. We start with the new Drinfeld realization of quantum affine algebras generalizing the loop realization of affine Lie algebras.

Let $U_q^R(\widehat{\mathfrak{g}})$ be an associative algebra generated by elements $X^{\pm}_{i,r}, r \in \mathbb{Z}$, $H_{i,r}, r \in \mathbb{Z}\backslash\{0\}$, $L_i^{\pm 1}, i = 1, \dots l$, $q^{\pm \frac{c}{2}}$. Put

$X_i^{\pm}(u) = \sum_{r\in\mathbb{Z}} X^{\pm}_{i,r} u^{-r}$,

$\Phi_i^{\pm}(u) = \sum_{r=0}^{\infty} \Phi^{\pm}_{i,\pm r} u^{\mp r} = K_i^{\pm 1} exp\left(\pm (q_i - q_i^{-1}) \sum_{s=1}^{\infty} H_{i,\pm s} u^{\mp s}\right)$,

$K_i = \prod_{j=1}^{l} L_j^{a_{ji}}$.

In terms of the generating series the defining relations are [3],[8]:

$$
\begin{aligned}
& L_i L_j = L_j L_i \ , \ L_i L_i^{-1} = L_i^{-1} L_i = 1, \\
& L_i X_j^{\pm}(u) L_i^{-1} = q_i^{\pm \delta_{i,j}} X_j^{\pm}(u), \\
& \Phi_i^{\pm}(u)\Phi_j^{\pm}(v) = \Phi_j^{\pm}(v)\Phi_i^{\pm}(u) \ , \ L_i \Phi_j^{\pm}(u) L_i^{-1} = \Phi_j^{\pm}(u), \\
& q^{\pm\frac{c}{2}} \text{ are central } , q^{\frac{c}{2}} q^{-\frac{c}{2}} = 1, \\
& \Phi_i^{+}(u)\Phi_j^{-}(v) = \frac{g_{ij}(\frac{vq^c}{u})}{g_{ij}(\frac{vq^{-c}}{u})} \Phi_j^{-}(v)\Phi_i^{+}(u), \\
& \Phi_i^{-}(u) X_j^{\pm}(v) \Phi_i^{-}(u)^{-1} = g_{ij}(\frac{uq^{\mp\frac{c}{2}}}{v})^{\pm 1} X_j^{\pm}(v), \\
& \Phi_i^{+}(u) X_j^{\pm}(v) \Phi_i^{+}(u)^{-1} = g_{ij}(\frac{vq^{\mp\frac{c}{2}}}{u})^{\mp 1} X_j^{\pm}(v), \\
& (u - vq^{\pm b_{ij}}) X_i^{\pm}(u) X_j^{\pm}(v) = (q^{\pm b_{ij}} u - v) X_j^{\pm}(v) X_i^{\pm}(u), \\
& X_i^{+}(u) X_j^{-}(v) - X_j^{-}(v) X_i^{+}(u) = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left(\delta(\frac{uq^{-c}}{v}) \Phi_i^{+}(vq^{\frac{c}{2}}) - \delta(\frac{uq^{c}}{v}) \Phi_i^{-}(uq^{\frac{c}{2}}) \right), \\
& \sum_{\pi\in S_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array} \right]_{q_i} \times \\
& X_i^{\pm}(z_{\pi(1)}) \dots X_i^{\pm}(z_{\pi(k)}) X_j^{\pm}(w) X_i^{\pm}(z_{\pi(k+1)}) \dots X_i^{\pm}(z_{\pi(1-a_{ij})}) = 0, \ i \neq j, \\
& \text{where } g_{ij}(z) = \frac{1 - q^{b_{ij}} z}{1 - q^{-b_{ij}} z} q^{-b_{ij}} \in \mathbb{C}[[z]].
\end{aligned}
\tag{22}
$$

The generators $X^{\pm}_{i,r}, H_{i,r}$ correspond to the elements $X^{\pm}_i z^r, H_i z^r$ of the affine Lie algebra $\widehat{\mathfrak{g}}$ in the loop realization (here $X^{\pm}_i, H_i$ are the Chevalley generators of $\mathfrak{g}$).

Let $I_n, n > 0$ be the left ideal in $U_q^R(\widehat{\mathfrak{g}})$ generated by $X^{\pm}_{i,r}, i = 1, \dots l, r \geq n$ and by all polynomials in $H_{i,r}, r > 0,\ L_i^{\pm 1}$ of degrees greater or equal to n $(deg(H_{i,r}) = r,\ deg(L_i^{\pm 1}) = 0)$. The algebra $\widehat{U}_q^R(\widehat{\mathfrak{g}})$,

$$\widehat{U}_q^R(\widehat{\mathfrak{g}}) = \lim_{\leftarrow} U_q^R(\widehat{\mathfrak{g}})/I_n \text{ (inverse limit) } .$$

is called the restricted completion of $U_q^R(\widehat{\mathfrak{g}})$. We fix $k \in \mathbb{C}$ and denote by $\widehat{U}_q^R(\widehat{\mathfrak{g}})_k$ the quotient of $\widehat{U}_q^R(\widehat{\mathfrak{g}})$ by the ideal generated by $q^{\pm \frac{c}{2}} - q^{\pm \frac{k}{2}}$. Sometimes it is convenient to use the weight–type generators $Y_{i,r}$,

$$Y_{i,r} = (a^r)^{-1}_{ik} H_{k,r},\ a^r_{ij} = \frac{1}{r}[r a_{ij}]_{q_i}.$$

In the spirit of theorem 4 we introduce, for any any set of complex numbers $n^{\pm r}_{ij}, i, j = 1, \dots l, r \in \mathbb{N}$ and integer parameters $n_{ij}, i, j = 1, \dots l$, generating series of the form,

$$e_i^{\{n\}}(u) = q_i^{-n_{ii}} \Phi_i^{0\{n\}} \Phi_i^-(u)^{\{n\}} X_i^+(u) \Phi_i^+(u)^{\{n\}}, \tag{23}$$

where

$$\begin{aligned} \Phi_i^{\pm}(u)^{\{n\}} &= exp\left(\textstyle\sum_{r=1}^{\infty} Y_{j,\pm r}\, n^{\pm r}_{ij} u^{\mp r}\right), n^{\pm r}_{ij} \in \mathbb{C}, \\ \Phi_i^{0\{n\}} &= \textstyle\prod_{j=1}^{l} L_j^{n_{ji}}, n_{ij} \in \mathbb{Z}. \end{aligned} \tag{24}$$

The fourier coefficients of $e_i^{\{n\}}(u)$ are elements of $\widehat{U}_q^R(\widehat{\mathfrak{g}})_k$.

Proposition 5 *The generating functions $e_i^{\{n\}}(u)$ satisfy the following commutation relations,*

$$\begin{aligned} &(u - vq^{b_{ij}}) F_{ji}(\tfrac{v}{u}) e_i^{\{n\}}(u) e_j^{\{n\}}(v) = (q^{b_{ij}} u - v) F_{ij}(\tfrac{u}{v}) e_j^{\{n\}}(v) e_i^{\{n\}}(u), \\ &\textstyle\sum_{\pi \in S_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} \prod_{p<q} F_{ii}(\frac{z_{\pi(q)}}{z_{\pi(p)}}) \prod_{r=1}^{k} F_{ji}(\frac{w}{z_{\pi(r)}}) \times \\ &\textstyle\prod_{s=k+1}^{1-a_{ij}} F_{ij}(\frac{z_{\pi(s)}}{w}) e_i^{\{n\}}(z_{\pi(1)}) \dots e_i^{\{n\}}(z_{\pi(k)}) e_j^{\{n\}}(w) \times \\ &e_i^{\{n\}}(z_{\pi(k+1)}) \dots e_i^{\{n\}}(z_{\pi(1-a_{ij})}) = 0,\ \mathit{for}\ i \neq j, \end{aligned} \tag{25}$$

where

$$F_{ij}(z) = q_j^{n_{ij}} exp(\sum_{r=1}^{\infty}((n_{ij}^{-r} - n_{ji}^{r})q^{-\frac{kr}{2}} - n_{ik}^{-r} n_{jl}^{r} r (B^r)^{-1}_{kl}(q^{kr} - q^{-kr}))z^r),$$

$$B_{ij}^r = q^{rb_{ij}} - q^{-rb_{ij}}. \tag{26}$$

Proof. Proposition is proved by direct substitution of the expression for the generating series $e_i^{\{n\}}$ and using the defining relations of the algebra $\widehat{U}_q^R(\widehat{\mathfrak{g}})_k$.

The commutation relations stated in proposition 5 are affine counterparts of the modified quantum Serre relations (6). We denote by $\widehat{F}_q^{\{n\}}$ the subalgebra in $\widehat{U}_q^R(\widehat{\mathfrak{g}})_k$ generated by Fourier coefficients of $e_i^{\{n\}}(u), i = 1, \dots l$.

Suppose that the map $\widehat{\chi}_q^{\{n\}} : \widehat{F}_q^{\{n\}} \to \mathbb{C}$ defined on the generators by equation $\widehat{\chi}_q^{\{n\}}(e_i^{\{n\}}(u)) = 1$ is a character of the algebra $\widehat{F}_q^{\{n\}}$. Then, relations (25) imply the following equations for the formal power series $F_{ij}(z)$:

$$(z - q^{b_{ij}})F_{ji}(z^{-1}) = (q^{b_{ij}}z - 1)F_{ij}(z), a_{ij} \neq 0, \tag{27}$$

$$F_{ji}(z^{-1}) = F_{ij}(z), a_{ij} = 0, \tag{28}$$

$$\sum_{\pi \in S_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{q_i} \prod_{p<q} F_{ii}(\tfrac{z_{\pi(q)}}{z_{\pi(p)}}) \prod_{r=1}^{k} F_{ji}(\tfrac{w}{z_{\pi(r)}}) \times \prod_{s=k+1}^{1-a_{ij}} F_{ij}(\tfrac{z_{\pi(s)}}{w}) = 0,\ a_{ij} \neq 0,\ i \neq j. \tag{29}$$

Our aim is to solve this system of equations with respect to parameters n_{ij}, n_{ij}^r. It is easy to find solutions of (27) and (28).

Proposition 6 *Suppose that $d_j n_{ij} = d_i n_{ji}$ for any i and j such that $a_{ij} = 0$. Then, the system of equations (27) and (28) has a unique solution in Taylor series, $\mathbb{C}[[z]]$, with constant terms $F_{ij}(0) = q_j^{n_{ij}}$. This solution is given by formula,*

$$F_{ij}(z) = \frac{q_j^{n_{ij}} - zq_i^{n_{ji}}}{1 - zq^{b_{ij}}},\ a_{ij} \neq 0, \tag{30}$$

$$F_{ij}(z) = q_j^{n_{ij}},\ a_{ij} = 0. \tag{31}$$

Note that parameters n_{ij} in proposition 6 may take arbitrary complex values. In applications to quantum groups n_{ij} are integers.

Proof. Suppose that the Taylor series

$$F_{ij}(z) = \sum_{n=0}^{\infty} c_{ij}^n z^n,$$

satisfy equations (27). Then, the r.h.s. of (27) are Taylor series. Therefore the l.h.s. of these equations must belong to $\mathbb{C}[[z]]$, $(z - q^{b_{ij}})F_{ji}(z^{-1}) \in \mathbb{C}[[z]]$. From the other hand $(F_{ji}(z^{-1}) - c_{ij}^0)(z - q^{b_{ij}}) \in \mathbb{C}[[z^{-1}]]$. It follows that $(F_{ji}(z^{-1}) - c_{ij}^0)(z - q^{b_{ij}}) = c_{ji} \in \mathbb{C}$, and

$$F_{ji}(z^{-1}) = c_{ij}^0 + c_{ij} \frac{z^{-1}}{1 - z^{-1} q^{b_{ij}}}. \tag{32}$$

Substituting this anzatz into (27) we get the following relations for the coefficients c_{ij}^0, c_{ij}:

$$c_{ij} = -c_{ji}^0 + q^{b_{ij}} c_{ij}^0.$$

Adding the condition $c_{ij}^0 = q_j^{n_{ij}}$, one obtains (30).

Equation (28) implies that $F_{ij}(z)$ is a constant for i, j such that $a_{ij} = 0$. Then, it is equal to the constant term of its Taylor series, which gives (31).

Next, we show that, under some assumptions, equations (27) and (28) imply (29).

Proposition 7 *Assume that $d_i n_{ji} - d_j n_{ij} = c_{ij}^{\pi}$ for some permutation $\pi \in S_l$. Then, any solution of (30) satisfies (29).*

Proof. We shall use theorem 10 and proposition 12 proved in Appendix.

An important property of solution (30) subject to the conditions of the proposition is that either $F_{ji} = q_i^{n_{ji}}$ or $F_{ij} = q_j^{n_{ij}}$. From this fact it follows that either the series in (40) or the series in (48) have a common domain of convergence. Therefore either in (40) or in (48) the product of formal power series is well defined. This allows us to apply theorem 10 or proposition 12, respectively, to obtain identities (29) for solution (30).

Next, for some choice of n_{ij} as in proposition 7 we would like to choose complex parameters $n_{ij}^r, r \neq 0$ such that the following eqation is satisfied,

$$(n_{ij}^{-r} - n_{ji}^r)q^{-\frac{kr}{2}} - n_{ik}^{-r} n_{jl}^r r (B^r)_{kl}^{-1}(q^{kr} - q^{-kr}) = \frac{1}{r}(q^{rb_{ij}} - q^{r(d_i n_{ji} - d_j n_{ij})}), r \in \mathbb{N}. \tag{33}$$

Equation (33) has many solutions. In particular, one can choose $n_{ij}^r = 0$ for all $r \leq 0$. Then,

$$n_{ji}^r = -q^{\frac{kr}{2}} \frac{1}{r}(q^{rb_{ij}} - q^{r(d_i n_{ji} - d_j n_{ij})})$$

for $r > 0$.

Finally, we conclude that if coefficients n_{ij} and n_{ij}^r are solutions of equations (18) and (33), the map $\widehat{\chi}_q^{\{n\}}$ is a character of the subalgebra $\widehat{F}_q^{\{n\}}$. Now we bring this result into focus by defining new realizations of affine quantum algebras, similar to the new realizations of finite–dimensional quantum algebras of the previous section.

First, we define quantum counterparts of $U(\mathfrak{n}((z)))$. Let E_q be the free associative algebra generated by Fourier coefficients of the generating series $e_i(u), i = 1, \ldots l$. Let $K_n, n > 0$ be the left ideal in E_q generated by $e_{i,r}, r \geq n$. Put $\widehat{E}_q = \lim_{\leftarrow} E_q/K_n$. Let $\widehat{F}_q^\pi$ be the quotient of $\widehat{E}_q$ by the two–sided ideal generated by the Fourier coefficients of the following generating series:

$$\begin{gathered}(u - vq^{b_{ij}})F_{ji}(\tfrac{v}{u})e_i(u)e_j(v) - (q^{b_{ij}}u - v)F_{ij}(\tfrac{u}{v})e_j(v)e_i(u), \\ \textstyle\sum_{\pi \in S_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array} \right]_{q_i} \prod_{p<q} F_{ii}(\frac{z_{\pi(q)}}{z_{\pi(p)}}) \prod_{r=1}^k F_{ji}(\frac{w}{z_{\pi(r)}}) \times \\ \textstyle\prod_{s=k+1}^{1-a_{ij}} F_{ij}(\frac{z_{\pi(s)}}{w}) e_i(z_{\pi(1)}) \ldots e_i(z_{\pi(k)}) e_j(w) e_i(z_{\pi(k+1)}) \ldots e_i(z_{\pi(1-a_{ij})}), i \neq j,\end{gathered} \tag{34}$$

where $F_{ij}(z)$ are given by (30),(31) .

The algebra $\widehat{F}_q^\pi$ is an abstract version of subalgebras $\widehat{F}_q^{\{n\}}$. Note that the defining relations of the algebra $\widehat{F}_q^\pi$ only depend on skew–symmetric combination (18) of the coefficients n_{ij}. Hence, similar to the finite-dimensional case, one can associate $\widehat{F}_q^\pi$ to a Coxeter element of the Weyl group.

Next, we define new realizations of affine quantum groups. Let A_q be the free associative algebra generated by the Fourier coefficients of generating series

$$
\begin{aligned}
&e_i(u) = \sum_{r\in\mathbb{Z}} e_{i,r}u^{-r},\\
&f_i(u) = \sum_{r\in\mathbb{Z}} f_{i,r}u^{-r},\\
&K_i^{\pm}(u) = \sum_{r=0}^{\infty} K_{i,\pm r}^{\pm}u^{\mp r},\\
&K_i^{\pm}(u)^{-1} = \sum_{r=0}^{\infty} {K_{i,\pm r}^{\pm}}^{-1}u^{\mp r}
\end{aligned}
\tag{35}
$$

and elements $L_i^{\pm 1}, i = 1, \ldots, l$.

Let $J_n, n > 0$ be the left ideal in A_q generated by $e_{i,r}, f_{i,r}, r \geq n$ and by all polynomials in $K_{i,r}^{+}, {K_{i,r}^{+}}^{-1}, L_i^{\pm 1}, r \geq 0$ of degrees greater than or equal to n ($deg({K_{i,r}^{+}}^{\pm 1}) = r, deg(L_i^{\pm 1}) = 0$). Put $\widehat{A}_q = \lim_{\leftarrow} A_q/J_n$.

For $F_{ij}(z)$ given by (30),(31) we define the following formal power series:

$$
\begin{aligned}
&M_{ij}(z) = g_{ij}(zq^{-k})^{-1}F_{ji}(zq^{k})F_{ji}(zq^{-k})^{-1},\\
&G_{ij}(z) = M_{ij}(zq^{-k})M_{ij}(zq^{k})^{-1},\\
&F_{ij}^{-}(z) = F_{ij}(zq^{2k}).
\end{aligned}
$$

Let $\widehat{U}_{q,k}^{\pi}(\widehat{\mathfrak{g}})$ be the quotient of $\widehat{A}_q$ by the two–sided ideal generated by the Fourier coefficients of the following generating series:

$$
\begin{aligned}
&K_i^{\pm}(u)K_j^{\pm}(v) - K_j^{\pm}(v)K_i^{\pm}(u)\ ,\ K_i^{\pm}(u)K_i^{\pm}(u)^{-1} - 1\ ,\ K_i^{\pm}(u)^{-1}K_i^{\pm}(u) - 1\ ,\\
&L_iL_j - L_jL_i\ ,\ L_iL_i^{-1} - 1\ ,\ L_i^{-1}L_i - 1\ ,\\
&L_iK_j^{\pm}(v)L_i^{-1} - K_j^{\pm}(v),\\
&K_{i,0}^{\pm} - (\textstyle\prod_{j=1}^{l} L_j^{a_{ji}})^{\pm 1},\\
&K_i^{+}(u)K_j^{-}(v) - G_{ij}(\tfrac{v}{u})K_j^{-}(v)K_i^{+}(u),
\end{aligned}
$$

$$
\begin{aligned}
&L_i e_j(u) L_i^{-1} - q_i^{\delta_{i,j}} e_j(u), \\
&L_i f_j(u) L_i^{-1} - q_i^{-\delta_{i,j}} f_j(u), \\
&K_i^+(u) e_j(v) - M_{ij}(\tfrac{v}{u}) e_j(v) K_i^+(u), \\
&K_i^+(u) f_j(v) - M_{ij}(\tfrac{vq^k}{u})^{-1} f_j(v) K_i^+(u), \\
&K_i^-(u) e_j(v) - M_{ji}(\tfrac{u}{v})^{-1} e_j(v) K_i^-(u), \\
&K_i^-(u) f_j(v) - M_{ji}(\tfrac{uq^k}{v}) f_j(v) K_i^-(u), \\
&(u - vq^{b_{ij}}) F_{ji}(\tfrac{v}{u}) e_i(u) e_j(v) - (q^{b_{ij}} u - v) F_{ij}(\tfrac{u}{v}) e_j(v) e_i(u), \\
&(u - vq^{-b_{ij}}) F_{ji}^-(\tfrac{v}{u}) f_i(u) f_j(v) - (q^{-b_{ij}} u - v) F_{ij}^-(\tfrac{u}{v}) f_j(v) f_i(u), \\
&\textstyle\sum_{\pi \in S_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array} \right]_{q_i} \prod_{p<q} F_{ii}(\frac{z_{\pi(q)}}{z_{\pi(p)}}) \prod_{r=1}^{k} F_{ji}(\frac{w}{z_{\pi(r)}}) \times \\
&\textstyle\prod_{s=k+1}^{1-a_{ij}} F_{ij}(\frac{z_{\pi(s)}}{w}) e_i(z_{\pi(1)}) \dots e_i(z_{\pi(k)}) e_j(w) e_i(z_{\pi(k+1)}) \dots e_i(z_{\pi(1-a_{ij})}), \ i \neq j, \\
&\textstyle\sum_{\pi \in S_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array} \right]_{q_i} \prod_{p<q} F_{ii}^-(\frac{z_{\pi(q)}}{z_{\pi(p)}}) \prod_{r=1}^{k} F_{ji}^-(\frac{w}{z_{\pi(r)}}) \times \\
&\textstyle\prod_{s=k+1}^{1-a_{ij}} F_{ij}^-(\frac{z_{\pi(s)}}{w}) f_i(z_{\pi(1)}) \dots f_i(z_{\pi(k)}) f_j(w) f_i(z_{\pi(k+1)}) \dots f_i(z_{\pi(1-a_{ij})}), \ i \neq j, \\
&q_i^{n_{ji}} F_{ji}^{-1}(\tfrac{vq^k}{u}) e_i(u) f_j(v) - q_i^{n_{ji}} F_{ij}^{-1}(\tfrac{uq^k}{v}) f_j(v) e_i(u) - \\
&- \tfrac{\delta_{i,j}}{q_i - q_i^{-1}} \left(\delta(\tfrac{uq^{-k}}{v}) K_i^+(v) - \delta(\tfrac{uq^k}{v}) K_i^-(v) \right).
\end{aligned}
\tag{36}
$$

These relations only depend on skew– symmetric combination (18) of the coefficients n_{ij} . Thus, there is a one–to–one correspondence between Coxeter elements s_π and the algebras $\widehat{U}_{q,k}^\pi(\widehat{\mathfrak{g}})$.

Of course, the algebra $\widehat{F}_q^\pi$ is a subalgebra of $\widehat{U}_{q,k}^\pi(\widehat{\mathfrak{g}})$, with respect to the natural embedding.

Finally, we show that $\widehat{U}_{q,k}^\pi(\widehat{\mathfrak{g}})$ is indeed a realization of $\widehat{U}_q^R(\widehat{\mathfrak{g}})_k$.

Proposition 8 *For every integer–valued solution of equation (18) and every solution of system (33) with* $d_i n_{ji} - d_j n_{ij} = c^{\pi}_{ij}$ *there exists an isomorphism of algebras* $\widehat{\psi}_{\{n\}} : \widehat{U}^{\pi}_{q,k}(\widehat{\mathfrak{g}}) \rightarrow \widehat{U}^{R}_{q}(\widehat{\mathfrak{g}})_k$ *given by :*

$$\widehat{\psi}_{\{n\}}(e_i(u)) = q_i^{-n_{ii}} \Phi_i^{0\{n\}} \Phi_i^-(u)^{\{n\}} X_i^+(u) \Phi_i^+(u)^{\{n\}},$$

$$\widehat{\psi}_{\{n\}}(f_i(u)) = \Phi_i^{0\{n\}^{-1}} \Phi_i^-(uq^k)^{\{n\}^{-1}} X_i^-(u) \Phi_i^+(uq^{-k})^{\{n\}^{-1}},$$

$$\widehat{\psi}_{\{n\}}(K_i^{\pm}(u)) = K_i^{\pm 1} exp(\textstyle\sum_{s=1}^{\infty} \pm (q_i - q_i^{-1}) H_{i,\pm s} q^{-\frac{sk}{2}} u^{\mp s} -$$

$$Y_{j,\pm s} n_{ij,\pm s}(q^{ks} - q^{-ks}) u^{\mp s}), \tag{37}$$

$$K_i = \textstyle\prod_{j=1}^{l} L_j^{a_{ji}},$$

$$\widehat{\psi}_{\{n\}}(L_i) = L_i,$$

where $\Phi_i^{0\{n\}}, \Phi_i^-(u)^{\{n\}}, \Phi_i^+(u)^{\{n\}}$ *are defined by (24).*

Proof. The proof is by straightforward verification of defining relations.

We shall identify $\widehat{F}^{\pi}_q$ with the subalgebra in $\widehat{U}^{\pi}_{q,k}(\widehat{\mathfrak{g}})$ generated by $e_{i,r}, r \in \mathbb{Z}$. Let $\widehat{\chi}^{\pi}_q$ be its canonical character , $\widehat{\chi}^{\pi}_q(e_i(u)) = 1$. For a fixed Coxeter element and for every solution of the corresponding equations (18) , (33) the isomorphism $\widehat{\psi}_{\{n\}}$ maps the subalgebra $\widehat{F}^{\pi}_q$ onto $\widehat{F}^{\{n\}}_q$. The algebra $\widehat{F}^{\pi}_q$ and the character $\widehat{\chi}^{\pi}_q$ may be regarded as well–defined quantum counterparts of the algebra $U(\mathfrak{n}((z)))$ and the character $\widehat{\chi}$, respectively. We shall call $\widehat{U}^{\pi}_{q,k}(\widehat{\mathfrak{g}})$ the Coxeter realization of the quantum group $\widehat{U}^{R}_{q}(\widehat{\mathfrak{g}})_k$ corresponding to s_{π}.

Finally observe that the map $\widehat{\chi}^{\pi}_{\varphi} : \widehat{F}^{\pi}_q \rightarrow \mathbb{C}$ defined by $\widehat{\chi}^{\pi}_{\varphi}(e_i(u)) = \varphi_i(u), i = 1, \ldots, l$, where $\varphi_i(u) \in \mathbb{C}((u))$ are arbitrary formal power series, is a character of the algebra $\widehat{F}^{\pi}_q$.

Appendix
A family of combinatorial identities

In [12] Jing proves the following identities for skew–symmetric polynomials.

Theorem 9 *([12])*

For any $m \in \mathbb{Z}, m \leq 0$ the following identity holds:

$$\sum_{\pi\in S_{1-m}}(-1)^{l(\pi)}\sum_{k=0}^{1-m}\begin{bmatrix} 1-m \\ k \end{bmatrix}_t \prod_{p<q}(z_{\pi(q)}-t^2 z_{\pi(p)})\times \prod_{r=1}^{k}(1-t^m\frac{z_{\pi(r)}}{w})\prod_{s=k+1}^{1-m}(\frac{z_{\pi(s)}}{w}-t^m)=0. \tag{38}$$

These identities were proved by looking at the representations of current algebras, the negative integers m arise as off-diagonal matrix entries a_{ij} of Cartan matrixes. Using Jing's identities (38) we will show that equations (27) imply identities (29). The latter are generalizations of Jing's identities.

Let $a_{ij}, i, j = 1, \ldots, l$ be a generalized Cartan matrix : $a_{ii} = 2$, a_{ij} are nonpositive integers for $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$. Suppose also that a_{ij} is symmetrizable , i. e. , there exist coprime positive integers $d_1, \ldots, d_l$ such that the matrix $b_{ij} = a_{ij}d_i$ is symmetric.

We shall make use of formal power series (f.p.s.) which are infinite in both directions. The space of such series is denoted by $\mathbb{C}((z))$. The product of two f.p.s. $f(z) = \sum_{n=-\infty}^{\infty} f_n z^n, g(z) = \sum_{n=-\infty}^{\infty} g_n z^n$ is said to exist if the coefficients of the series

$$\sum_{p=-\infty}^{\infty} z^p \sum_{k+n=p} f_n g_k$$

are well defined , i.e. the series $\sum_{k+n=p} f_n g_k$ converges for every p. Similarly , the product of three f.p.s $f(z) = \sum_{n=-\infty}^{\infty} f_n z^n$, $g(z) = \sum_{n=-\infty}^{\infty} g_n z^n$, $h(z) = \sum_{n=-\infty}^{\infty} h_n z^n$ exists if the series $\sum_{k+n+l=p} f_n g_k h_l$ converges for every p and its sum does not depend on the ordering of the terms. Clearly , in this case the products $g(z)h(z), g(z)f(z)$ and $f(z)h(z)$ are well–defined. For instance , if two or more formal power series have a common domain of convergence their product is well–defined. We often use notation $\frac{1}{1-x}$ for the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

viewed as a formal power series.

We show that equations (27):

$$(z-q^{b_{ij}})F_{ji}(z^{-1}) = (q^{b_{ij}}z-1)F_{ij}(z),\ a_{ij}\neq 0. \tag{39}$$

imply the system of identities (29) for formal power series. Note that these identities hold true not only for the Taylor series solution of (39).

Theorem 10 *Let $F_{kl}(z), k, l = 1, \ldots, l$ be a solution of equations (39). Suppose that for some i and j, $i \neq j$, the following product is well-defined as a formal power series,*

$$\prod_{p\neq q} \frac{1}{1 - q^{b_{ii}} \frac{z_q}{z_p}} \cdot \prod_{s=1}^{1-a_{ij}} \frac{1}{1 - q^{b_{ij}} \frac{z_s}{w}} \cdot P_{ij}, \tag{40}$$

where

$$P_{ij} = \sum_{\pi \in S_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{q_i} \prod_{p<q} F_{ii}(\frac{z_{\pi(q)}}{z_{\pi(p)}}) \prod_{r=1}^{k} F_{ji}(\frac{w}{z_{\pi(r)}}) \prod_{s=k+1}^{1-a_{ij}} F_{ij}(\frac{z_{\pi(s)}}{w}). \tag{41}$$

Then, $P_{ij} = 0$, as a formal power series.

The l.h.s of (41) is symmetric with respect to permutations of the formal variables $z_1, \ldots, z_{1-a_{ij}}$. Thus, the last theorem yields a family of combinatorial identities for symmetric functions.

Now we turn to the proof of the theorem. First, we prove the following lemma.

Lemma 11 *Let $F_{kl}(z), k, l = 1, \ldots, l$ be a solution of system (39). Then*

$$\prod_{p\neq q} (1 - q^{b_{ii}} \frac{z_q}{z_p}) \prod_{s=1}^{1-a_{ij}} (1 - q^{b_{ij}} \frac{z_s}{w}) P_{ij} = 0, \ \text{for } i \neq j. \tag{42}$$

Proof. Let $\pi \in S_{1-a_{ij}}$. Consider the product:

$$\prod_{s=1}^{1-a_{ij}} (1 - q^{b_{ij}} \frac{z_s}{w}) \prod_{r=1}^{k} F_{ji}(\frac{w}{z_{\pi(r)}}) \prod_{s=k+1}^{1-a_{ij}} F_{ij}(\frac{z_{\pi(s)}}{w}).$$

The f.p.s. $\prod_{s=1}^{1-a_{ij}} (1 - q^{b_{ij}} \frac{z_s}{w})$ is symmetric with respect to permutations of the formal variables z_s. Therefore

$$\prod_{s=1}^{1-a_{ij}}(1-q^{b_{ij}}\frac{z_s}{w})\prod_{r=1}^{k}F_{ji}(\frac{w}{z_{\pi(r)}})\prod_{s=k+1}^{1-a_{ij}}F_{ij}(\frac{z_{\pi(s)}}{w}) =$$

$$\prod_{s=1}^{1-a_{ij}}(1-q^{b_{ij}}\frac{z_\pi(s)}{w})\prod_{r=1}^{k}F_{ji}(\frac{w}{z_{\pi(r)}})\prod_{s=k+1}^{1-a_{ij}}F_{ij}(\frac{z_{\pi(s)}}{w}).$$

Now using equations (39) for F_{ij} we obtain:

$$\prod_{s=1}^{1-a_{ij}}(1-q^{b_{ij}}\frac{z_\pi(s)}{w})\prod_{r=1}^{k}F_{ji}(\frac{w}{z_{\pi(r)}})\prod_{s=k+1}^{1-a_{ij}}F_{ij}(\frac{z_{\pi(s)}}{w}) =$$

$$\prod_{s=1}^{k}(1-q^{b_{ij}}\frac{z_{\pi(s)}}{w})\prod_{s=k+1}^{1-a_{ij}}(q^{b_{ij}}-\frac{z_{\pi(s)}}{w})\prod_{r=1}^{1-a_{ij}}F_{ji}(\frac{w}{z_{\pi(r)}}) =$$

$$\prod_{s=1}^{k}(1-q^{b_{ij}}\frac{z_{\pi(s)}}{w})\prod_{s=k+1}^{1-a_{ij}}(q^{b_{ij}}-\frac{z_{\pi(s)}}{w})\prod_{r=1}^{1-a_{ij}}F_{ji}(\frac{w}{z_r}), \tag{43}$$

since $\prod_{r=1}^{1-a_{ij}}F_{ji}(\frac{w}{z_{\pi(r)}})$ is also a symmetric f.p.s..
Similarly,

$$\begin{aligned}&\textstyle\prod_{p\neq q}(1-q^{b_{ii}}\frac{z_q}{z_p})\prod_{p<q}F_{ii}(\frac{z_{\pi(q)}}{z_{\pi(p)}}) =\\ &\textstyle\prod_{p>q}\left(\frac{1}{z_q}(1-q^{b_{ii}}\frac{z_q}{z_p})F_{ii}(\frac{z_q}{z_p})\right)\prod_{p<q}(-1)^{l(\pi)}(z_{\pi(q)}-q^{b_{ii}}z_{\pi(p)}).\end{aligned} \tag{44}$$

Substituting (43) and (44) into (42) we get :

$$\prod_{p\neq q}(1-q^{b_{ii}}\frac{z_q}{z_p})\prod_{s=1}^{1-a_{ij}}(1-q^{b_{ij}}\frac{z_s}{w})P_{ij} = \tag{45}$$

$$\prod_{r=1}^{1-a_{ij}} F_{ji}(\tfrac{w}{z_r}) \prod_{p>q} \left(\tfrac{1}{z_q}(1-q^{b_{ii}}\tfrac{z_q}{z_p})F_{ii}(\tfrac{z_q}{z_p})\right) \times$$

$$\sum_{\pi\in S_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array}\right]_{q_i} \times \tag{46}$$

$$\prod_{p<q}(-1)^{l(\pi)}(z_{\pi(q)}-q^{b_{ii}}z_{\pi(p)})\times$$

$$\prod_{s=1}^{k}(1-q^{b_{ij}}\tfrac{z_{\pi(s)}}{w})\prod_{s=k+1}^{1-a_{ij}}(q^{b_{ij}}-\tfrac{z_{\pi(s)}}{w}).$$

Now lemma 11 follows immediately from theorem 9 with $t=q_i,\ m=a_{ij}$.

Proof of the theorem. The conditions of the theorem imply that, as a formal power series,

$$P_{ij}=\prod_{p\neq q}\frac{1}{1-q^{b_{ii}}\frac{z_q}{z_p}}\prod_{s=1}^{1-a_{ij}}\frac{1}{1-q^{b_{ij}}\frac{z_s}{w}}\prod_{p\neq q}(1-q^{b_{ii}}\frac{z_q}{z_p})\prod_{s=1}^{1-a_{ij}}(1-q^{b_{ij}}\frac{z_s}{w})P_{ij}. \tag{47}$$

From lemma 11 it follows that

$$\prod_{p\neq q}(1-q^{b_{ii}}\frac{z_q}{z_p})\prod_{s=1}^{1-a_{ij}}(1-q^{b_{ij}}\frac{z_s}{w})P_{ij}=0.$$

Therefore, $P_{ij}=0$. This concludes the proof.

One can formulate several versions of theorem 10. For instance, the following is true.

Proposition 12 *Let $F_{kl}(z), k,l=1,\ldots,l$ be a solution of system (39). Suppose that for some i and j the product*

$$\prod_{p\neq q}\frac{1}{1-q^{b_{ii}}\frac{z_q}{z_p}}\cdot\prod_{1}^{1-a_{ij}}\frac{\frac{w}{z_s}}{1-q^{b_{ij}}\frac{w}{z_s}}\cdot P_{ij} \tag{48}$$

is well–defined as a formal power series. Then, $P_{ij}=0$.

Proof of the proposition is similar to that of theorem 10.

Similar statements exist for $|q|>1$.

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